Jonathan Wand, PS350c Handout, April 21 2004

1 Nested logit

1.1 With independence throughout

Let there be an index of groups of choices $j \in \{1, \ldots, J\}$, and within each group there is an index of specific alternatives $k \in \{1, \ldots, K_j\}$. kj represents choice k within group j, and $k|j$ be the choice of j conditional on being in group k.

Let the utility of choice jk be $U_{jk} = \mu_{jk} + \nu_j + \epsilon_{jk}$, where ϵ_{jk} is distributed iid Gumbel. Consider the choice probabilities conditional on group,

$$P_{k|j} = \frac{e^{U_{jk}}}{\sum_j^{K_j} e^{U_{jk}}} = \frac{e^{\mu_{jk}}}{\sum_k^{K_j} e^{\mu_{jk}}} = \frac{e^{\mu_{jk}}}{e^{I_j}} \tag{1}$$

and the group probabilities,

$$P_j = \frac{\sum_k^{K_j} e^{U_{jk}}}{\sum_{j'}^J \sum_k^{K_{j'}} e^{U_{j'k}}} = \frac{e^{\nu_j} \sum_k^{K_j} e^{\mu_{jk}}}{\sum_{j'}^J e^{\nu_{j'}} \sum_k^{K_{j'}} e^{\mu_{j'k}}} = \frac{e^{\nu_j + I_j}}{\sum_{j'}^J e^{\nu_{j'} + I_{j'}}} \tag{2}$$

where,

$$I_j = \log\left(\sum_k^{K_j} e^{\mu_{jk}}\right) \tag{3}$$

so,

$$P_{jk} = P_{k|j} P_j = \frac{e^{\mu_{jk}}}{e^{I_j}} \frac{e^{\nu_j + I_j}}{\sum_{j'}^J e^{\nu_{j'} + I_{j'}}} \tag{4}$$

1.2 With correlation within nests

Now, allow for the possibility that the errors within nests (but not across nests) are correlated. A multivariate version of the Gumbel distribution (Kotz, Blakrishnan, and Johnston, 2000, 642–3) is

$$G(\epsilon_{j1}, \ldots, \epsilon_{jK_j}) = \exp\left\{ - \left(\sum_k^{K_j} \exp\{-\epsilon_{jk}/\rho_j\} \right)^{\rho_j} \right\} \tag{5}$$

Let's write this out for the bivariate case,

$$G(\epsilon_{j1}, \epsilon_{j2}) = \exp\left\{ - \left(\exp\{-\epsilon_{j1}/\rho_j\} + \exp\{-\epsilon_{j2}/\rho_j\} \right)^{\rho_j} \right\} \tag{6}$$

and note that if $\rho_j = 1$, we have independence,

$$\begin{aligned} G(\epsilon_{j1}, \epsilon_{j2}) &= \exp\left\{ - \left(e^{-\epsilon_{j1}} + e^{-\epsilon_{j2}} \right) \right\} && (7) \\ &= \exp\left\{ -e^{-\epsilon_{j1}} \right\} \exp\left\{ -e^{-\epsilon_{j2}} \right\} = \Lambda_j(\epsilon_{j1})\Lambda_j(\epsilon_{j2}) && (8) \end{aligned}$$

By integration of discriminal processes,

$$P(U_{jk} > U_{jk'}, \forall k \neq k') \tag{9}$$

with G instead of independent Gumbels, we get

$$P_{k|j} = \frac{e^{U_{jk}/\rho_j}}{\sum_k^{K_j} e^{U_{jk}/\rho_j}} = \frac{e^{\mu_{jk}/\rho_j}}{\sum_k^{K_j} e^{\mu_{jk}/\rho_j}} = \frac{e^{\mu_{jk}/\rho_j}}{e^{I'_j}} \tag{10}$$

where,

$$I_j = \log\left(\sum_k^{K_j} e^{\mu_{jk}/\rho_j} \right) \tag{11}$$

and the group probabilities,

$$P_j = \frac{\left(\sum_k^{K_j} e^{U_{jk}/\rho_j} \right)^{\rho_j}}{\sum_{j'}^{J} \left(\sum_k^{K_{j'}} e^{U_{j'k}/\rho_{j'}} \right)^{\rho_{j'}}} = \frac{e^{\nu_j} \left(\sum_k^{K_j} e^{\mu_{jk}/\rho_j} \right)^{\rho_j}}{\sum_{j'}^{J} e^{\nu_{j'}} \left(\sum_k^{K_{j'}} e^{\mu_{j'k}/\rho_{j'}} \right)^{\rho_{j'}}} = \frac{e^{\nu_j + \rho I_j}}{\sum_{j'}^{J} e^{\nu_{j'} + \rho I'_{j'}}} \tag{12}$$

and Train (p.90) shows how to get use these results to go back and forth to an alternate arrangment,

$$P_{jk} = P_{k|j} P_j = \frac{e^{\mu_{jk}/\rho_j}}{e^{I'_j}} \frac{e^{\nu_j + \rho I_j}}{\sum_{j'}^{J} e^{\nu_{j'} + \rho I'_{j'}}} = \frac{e^{U_{jk}/\rho_j} \left(\sum_{k'}^{K_j} e^{U_{jk'}/\rho_j} \right)^{1-\rho_j}}{\sum_{j'}^{J} \left(\sum_{k'}^{K_{j'}} e^{U_{j'k'}/\rho_{j'}} \right)^{\rho_{j'}}} \tag{13}$$

if $\rho = 1$, and therefore independent, we get back equation (2).

1.3 Mebane (2000) with partial nesting

Two types/groups of voting choices, $J \in \{S, Z\}$

1. S: straight-ticket (DD, RR)
2. Z: split-ticket (DR, RD)

Mebane specifies a model where straight-ticket utilities are indepdent ($\rho_s = 1$) but split-ticket utilities are correlated $0 < \rho_z < 1$.

$$P(DD,S) = P_{DD|S}P(S) = \frac{e^{\mu_{DD}/\rho_s}}{e^{I_s'}} \frac{e^{\nu_s+\rho I_s}}{e^{\nu_s+\rho I_s} + e^{\nu_z+\rho I_z}} \tag{14}$$

$$= \frac{e^{U_{DD}/\rho_s}\left(e^{U_{DD}/\rho_s} + e^{U_{RR}/\rho_s}\right)^{1-\rho_s}}{\left(e^{U_{DD}/\rho_s} + e^{U_{RR}/\rho_s}\right)^{\rho_s} + \left(e^{U_{RD}/\rho_z} + e^{U_{DR}/\rho_z}\right)^{\rho_z}} \tag{15}$$

$$= \frac{e^{U_{DD}}}{e^{U_{DD}} + e^{U_{RR}} + \left(e^{U_{RD}/\rho_z} + e^{U_{DR}/\rho_z}\right)^{\rho_z}} \tag{16}$$

$$P(RR,S) = P_{RR|S}P(S) = \frac{e^{U_{RR}}}{e^{U_{DD}} + e^{U_{RR}} + \left(e^{U_{RD}/\rho_z} + e^{U_{DR}/\rho_z}\right)^{\rho_z}} \tag{17}$$

$$P(DR,Z) = P_{DR|Z}P(Z) = \frac{e^{U_{DR}/\rho_z}\left(e^{U_{RD}/\rho_z} + e^{U_{DR}/\rho_z}\right)^{1-\rho_z}}{e^{U_{DD}} + e^{U_{RR}} + \left(e^{U_{RD}/\rho_z} + e^{U_{DR}/\rho_z}\right)^{\rho_z}} \tag{18}$$

$$P(RD,Z) = P_{RD|Z}P(Z) = \frac{e^{U_{RD}/\rho_z}\left(e^{U_{RD}/\rho_z} + e^{U_{DR}/\rho_z}\right)^{1-\rho_z}}{e^{U_{DD}} + e^{U_{RR}} + \left(e^{U_{RD}/\rho_z} + e^{U_{DR}/\rho_z}\right)^{\rho_z}} \tag{19}$$

$$\tag{20}$$

This gets us a more explicit set of probabilities described on page 42 of Mebane (2000).

2 Absention due to indifference, Sanders (1997)

Three choices: vote D, vote R, or Abstain. Unlike other choices, Sanders builds on spatial theory of voting which posits that abstention is result of indifference between parties (e.g., Munger and Hinich, Analytical Politics). For $T \geq 0$,

1. vote D if $U_D - U_R > T$
2. vote R if $U_R - U_D > T$
3. abstain if $-T < U_D - U_R < T$

$$
\begin{aligned}
P(D) &= P[(\mu_D + \epsilon_D) - (\mu_R + \epsilon_R) > T] && (21)\\
&= P[(\mu_D + \epsilon_D) - \mu_R - T > \epsilon_R] && (22)\\
&= \int_{-\infty}^{\infty} f(\epsilon_D) \int^{\mu_D + \epsilon_D - \mu_R - T} f(\epsilon_R) \partial \epsilon_R \partial \epsilon_D && (23)\\
&= \frac{1}{1 + \exp\{-(\mu_D - \mu_R) e^T\}} && (24)
\end{aligned}
$$

Extension: Munger and Hinich, within same spatial theory of voting, also posit alienation. Could modify to have upper bound on absolute distance, or lower bound on absolute utility, of each party.